Partial Chromatic Polynomials and Diagonally Distinct Sudoku Squares

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1 Introduction

This paper is based on a talk I gave at Illinois State University on April 10, 2008, and contains two proofs. The first one is of a statement about completions of partial λ -colorings of a graph in a very interesting article by Herzberg and Murty [2], namely, the fact that the number of possible completions is a polynomial in λ (which we will call the partial chromatic polynomial). Two elegant proofs of this statement, one with Möbius inversion and the other by induction, are already given in [2]: both proofs use the concept of contraction. Our alternative proof mimics the construction of the classical chromatic polynomial instead. The second proof in this paper shows that there exist $n^2 \times n^2$ Sudoku squares with distinct entries in both diagonals (in addition to the rows, columns, and $n \times n$ sub-grids) for all n. I would like to thank Walter "Wal" Wallis for pointing out (days after I posted the proof and gave the talk) that there exists an earlier and very similar proof of the existence of such squares, due to A.D. Keedwell [3]: I was unaware of [3] at the time. I would also like to thank Papa Sissokho for correcting my terminology and making the first proof more palatable.

2 Partial chromatic polynomial

Sudoku puzzles are, in a discrete mathematician's world, partially colored graphs. Questions about the minimal number of clues for unique solutions etc. boil down to questions about partial colorings of the "Sudoku graph". This graph consists of n^4 vertices, corresponding to the squares of an $n^2 \times n^2$ Sudoku grid, such that any two distinct vertices in the same row, column, or sub-grid are joined by an edge. A completed Sudoku puzzle is then a proper coloring of the Sudoku graph with n^2 colors.

Theorem 1. [2] Let G be a graph with n vertices, and C be a partial proper coloring of t vertices of G using exactly λ_0 colors. Define $p_{G,C}(\lambda)$ to be the

number of ways C can be completed to a proper λ -coloring of G. Then for $\lambda \geq \lambda_0$, the expression $p_{G,C}(\lambda)$ is a monic polynomial in λ of degree n-t.

Proof. Let C be a partial proper coloring of t vertices of G with exactly λ_0 colors. Call a proper coloring C' of G "consistent with C" if the vertices colored under C keep their colors under C'. Also call a proper coloring C' "generic" if it is simply a partitioning of the vertices of G into independent sets (more precisely, a generic coloring is an equivalence class of colorings with the same independent sets). Now let C' be any generic proper coloring of G with exactly λ_0 independent sets. If C' is consistent with C, then there is only 1 way the colors of C' can be specified; the larger independent sets in C' have to retain the colors of the smaller ones in C. Next, if a generic C' is to be consistent with C and have $\lambda_0 + 1$ independent sets, then there are $(\lambda - \lambda_0)$ ways of specifying the colors of C': for the λ_0 sets that extend those in C, we have no choice but to respect the colors dictated by C. On the other hand, the extra independent set does not intersect C, so we can use any of the remaining $(\lambda - \lambda_0)$ colors. We continue the argument for all generic proper colorings with exactly $\lambda_0 + r$ independent sets, where $0 \le r \le n - t$. In short, we have

$$p_{G,C}(\lambda) = \sum_{r=0}^{n-t} m_r(G,C)(\lambda - \lambda_0) \cdots (\lambda - \lambda_0 - r + 1).$$

Here $m_r(G,C)$ is the number of generic proper colorings C' of G that are consistent with C and have exactly $\lambda_0 + r$ independent sets, and $(\lambda - \lambda_0) \cdots (\lambda - \lambda_0 - r + 1)$ is the number of ways the colors of such C' can be specified. The r-th term of the sum is a polynomial of degree r, and the (n-t)-th term is monic, because there is only one generic C' that adds n-t independent sets to C. Namely, each vertex outside C is a set by itself.

3 Diagonally distinct Sudoku squares

The existence of $n^2 \times n^2$ Sudoku squares for any positive integer n is a well-known fact (see [2] for a proof). We will show that it is moreover possible to construct $n^2 \times n^2$ Sudoku squares with distinct entries on each of the two diagonals for any n. A similar proof was given earlier, and unknown to the author at the time of e-publication of the first version of this paper, by Keedwell [3]. Michalowski et al. [4] and Bailey et al. [1] give some motivating real-life examples for variations of Sudoku puzzles and other gerechte designs.

Theorem 2. There exist $n^2 \times n^2$ Sudoku squares with distinct entries in the two diagonals, in addition to distinct entries in each row, column, and $n \times n$ sub-grid.

Proof. Notation: the (i, j)-block will be the $n \times n$ sub-grid placed according to matrix-entry enumeration convention inside the full grid (ith from the top and ith from the left). We will also enumerate entries in any block by the row and

column numbers in the block; thus, the (r, c)-entry of the (i, j)-block will be the (r + (i - 1)n, c + (j - 1)n)-entry of the complete grid. When writing indices, we will always choose the least positive residue modulo n (denoted by [x] for any integer x). As a result, all variables i, j, r, c, [x] will have values in the set $\{1, \ldots, n\}$.

Let the symbols a(r,c), with $1 \le r,c \le n$, denote the n^2 integers from 1 to n^2 in some order. We place these distinct integers in the upper left $n \times n$ block of the grid, now called the (1,1)-block, such that a(r,c) is in row r and column c:

a(1,1)	a(1,2)	a(1,3)		1	2	3
a(2,1)	a(2,2)	a(2,3)	=	4	5	6
a(3,1)	a(3,2)	a(3,3)		7	8	9

In order to create the (1,2)-block, we simply move the rows of the (1,1)-block up in a cyclic fashion:

Ī	a(2,1)	a(2,2)	a(2,3)	4	5	6
		a(3,2)		7	8	9
	a(1,1)		a(1,3)	1	2	3

We continue this permutation of rows inside each new block until we finish the first row of blocks. As for the (2,1)-block, we advance the rows inside the (1,1)-block one step down cyclically, and also move the entries in each row (inside the block) one step backward:

a(3,2)	a(3,3)	a(3,1)		8	9	7
a(1,2)	a(1,3)	a(1,1)	=	2	3	1
a(2,2)	a(2,3)	a(2,1)		5	6	4

We complete the second row of blocks similar to the first, only by permuting whole rows in the (2,1)-block upward, without making any changes to the rows internally, and repeat the procedure until all rows of blocks are exhausted. The 4×4 , 9×9 , and 16×16 Sudoku squares with distinct diagonal entries constructed by this method are given below:

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
8	9	7	2	3	1	5	6	4
2	3	1	5	6	4	8	9	7
5	6	4	8	9	7	2	3	1
6	4	5	9	7	8	3	1	2
9	7	8	3	1	2	6	4	5
3	1	2	6	4	5	9	7	8

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
5	6	7	8	9	10	11	12	13	14	15	16	1	2	3	4
9	10	11	12	13	14	15	16	1	2	3	4	5	6	7	8
13	14	15	16	1	2	3	4	5	6	7	8	9	10	11	12
14	15	16	13	2	3	4	1	6	7	8	5	10	11	12	9
2	3	4	1	6	7	8	5	10	11	12	9	14	15	16	13
6	7	8	5	10	11	12	9	14	15	16	13	2	3	4	1
10	11	12	9	14	15	16	13	2	3	4	1	6	7	8	5
11	12	9	10	15	16	13	14	3	4	1	2	7	8	5	6
15	16	13	14	3	4	1	2	7	8	5	6	11	12	9	10
3	4	1	2	7	8	5	6	11	12	9	10	15	16	13	14
7	8	5	6	11	12	9	10	15	16	13	14	3	4	1	2
8	5	6	7	12	9	10	11	16	13	14	15	4	1	2	3
12	9	10	11	16	13	14	15	4	1	2	3	8	5	6	7
16	13	14	15	4	1	2	3	8	5	6	7	12	9	10	11
4	1	2	3	8	5	6	7	12	9	10	11	16	13	14	15

We now present the full proof of existence. Let us place the integer

$$a\left(\left[\,r-(i-1)+(j-1)\,\right],\left[\,c+(i-1)\,\right]\,\right)=a\left(\left[\,r-i+j\right],\left[\,c+i-1\right]\,\right)$$

in the (r,c)-entry of the (i,j)-block, and use the prime notation to distinguish another entry. Distinct entries in the same row of the full grid (where i=i' and r=r', but $j\neq j'$ or $c\neq c'$) are not equal: if they were, then we would have

$$[r-i+j] = [r-i+j']$$
 and $[c+i-1] = [c'+i-1]$
 $\Rightarrow j=j'$ and $c=c'$.

Similarly, distinct entries in the same column of the full matrix (where j=j' and c=c', but $i\neq i'$ or $r\neq r'$) cannot be equal:

$$[r-i+j] = [r'-i'+j] \text{ and } [c+i-1] = [c+i'-1]$$

$$\Rightarrow \quad i=i' \text{ and } r=r'.$$

Two distinct entries in the same block (where i = i' and j = j', but $r \neq r'$ or $c \neq c'$) are not equal:

$$[r-i+j] = [r'-i+j]$$
 and $[c+i-1] = [c'+i-1]$
 $\Rightarrow r = r'$ and $c = c'$.

Two distinct entries on the main diagonal (where $i=j,\ i'=j',\ r=c,$ and r'=c', but $i\neq i'$ or $r\neq r'$) are not equal:

$$[r-i+i] = [r'-i'+i']$$
 and $[r+i-1] = [r'+i'-1]$
 $\Rightarrow r = r'$ and $i = i'$.

Finally, two distinct entries on the secondary diagonal (where i+j=i'+j'=n+1, r+c=r'+c'=n+1, but $i\neq i'$ or $r\neq r'$) are not equal:

$$[r-i+(n+1)-i] = [r'-i'+(n+1)-i']$$
and $[(n+1)-r+i-1] = [(n+1)-r'+i'-1]$

$$\Rightarrow [r-2i] = [r'-2i'] \text{ and } [-r+i] = [-r'+i']$$

$$\Rightarrow r = r' \text{ and } i = i'.$$

Calculations of the symmetries, the number of essentially different squares, the minimum number of entries in a puzzle to assure a unique solution, the asymptotic values of related expressions, and the partial or full chromatic polynomials for Sudoku graphs of rank n, are mentioned in [2] in relation to standard Sudoku squares. Similar calculations would certainly be interesting for the diagonally distinct $n^2 \times n^2$ Sudoku squares.

References

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- [4] M. Michalowski, C.A. Knoblock, and B.Y. Choueiry, http://consystlab.unl.edu/our*work/Papers/MichalowskiCPWS07.pdf (insert the underscore character for *)